HUNTING FOR CURVES WITH MANY POINTS

GERARD VAN DER GEER

Abstract. We construct curves with many points over finite fields by using the class group.

1. Introduction

The question how many rational points a curve of given genus over a finite field of given cardinality can have is an attractive challenge and the table of curves with many points ([10]) has been a reference point for progress for genus ≤ 50 and small fields of characteristic $p = 2$ and $p = 3$. The tables record there for a pair (q, g) an interval $[a, b]$ where b is the best upper bound for the maximum number of points of a curve of genus g over $\mathbb{F}_q$ and a gives a lower bound obtained from an explicit example of a curve C defined over $\mathbb{F}_q$ with a (or at least a) rational points. So $N_q(g) \in [a, b]$ with $N_q(g)$ the maximum number of rational points on a smooth connected projective curve defined over $\mathbb{F}_q$. For progress on the upper bounds we refer to the papers of Howe and Lauter, [1] and to the references given in the tables. It is the purpose of this little paper to record recent improvements of the table and at the same time give constructions for many of the present records of the table. The methods employ the class group and are variations on well-known themes and do not involve new ideas. We hope it inspires people to take up the challenge to improve the tables.

2. Using the Class Group

Let C be a curve of genus g defined over $\mathbb{F}_2$ with 'many' rational points. If f is a rational function not of the form $h^2 + h$ for h in the function field $\mathbb{F}_2(C)$ of C then in the Artin-Schreier cover of C defined by $w^2 + w = f$ all the rational points of C where f vanishes will be split. If the contribution of the poles of f is limited and f vanishes in many points this may yield curves over $\mathbb{F}_2$ with many points. Serre applied this method (cf. [7]) to construct curves over $\mathbb{F}_2$ of genus $g = 11$ with 14 rational points (resp. $g = 13$ with $N = 15$ and $g = 10$ with $N = 13$) thus determining $N_2(10) = 13$, $N_2(11) = 14$ and $N_2(13) = 15$. To find suitable f one uses the class group. Serre used an elliptic curve, but we may of course use higher genus curves as well. We illustrate the method by constructing many curves that reach or improve the present 'records'.

2.1. **Base Curve of Genus 1 over $\mathbb{F}_2$.** Let C be the (projective smooth) curve of genus 1 defined by the affine equation $y^2 + y = x^3 + x$. It has five rational points $P_0 = \infty$, $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (1, 1)$ and $P_4 = (0, 1)$. Since we have an isomorphism $C(\mathbb{F}_2) \cong \mathbb{Z}/5\mathbb{Z}$ of abelian groups such that under the isomorphism P_i

1991 *Mathematics Subject Classification.* 14K10.

corresponds to $i \pmod 5$, the divisor $\sum a_i P_i$ is of degree 0 and linearly equivalent to 0 if and only if

$$\sum_{i=0}^{4} a_i = 0 \quad \text{and} \quad \sum_{i=0}^{4} a_i\, i \equiv 0 \pmod 5. \tag{1}$$

So there exists a function $f \in \mathbb{F}_2(C)$ with divisor (f) equal to a given divisor $\sum a_i P_i$ if and only if the 5-tuple $(a_0, a_1, \ldots, a_4) \in \mathbb{Z}^5$ satisfies (1).

Often we shall denote a relation for linear equivalence $\sum_{i=0}^{4} a_i P_i \sim 0$ simply by $[a_0, a_1, \ldots, a_4]$ for shortness. Two examples of such relations for the present curve are $[-3, -1, 2, 1, 1]$ and $[-1, -3, 1, 1, 2]$ with functions f_1 and f_2. This gives curves C_{f_1} and C_{f_2} of genera $g = 4$ with $N = 8$ rational points. Note that $N_2(4) = 8$, so these curves realize the maximum for genus 4. The fibre product $C_{f_1} \times_C C_{f_2}$ has genus $g = 11$ with $N = 14$ rational points. Since the upper bound for $N_2(11)$ is 14, Serre (cf. [7]) thus showed that $N_2(11) = 14$.

Another example is the relation $-7P_0 + 2P_1 + 3P_2 + P_4 + P_5 \sim 0$. If f_3 is a corresponding function then the Artin-Schreier cover is a curve C_{f_3} with genus $g = 5$ and with $N = 9$ rational points as one readily checks. Note that $N_2(5) = 9$. The fibre product $C_{f_1} \times_C C_{f_3}$ is a curve of genus $g = 13$ with $N = 15$ rational points, hence $N_2(13) = 15$, again due to Serre.

Combining f_1 with the function f_4 corresponding to the relation $[-3, 2, 1, -1, 1]$ gives $C_{f_1} \times_C C_{f_4}$ of genus 10 with $N = 13$ rational points. Note that the divisor of $f_1 + f_4$ is $-2P_0 - P_1 + P_2 + 2P_4 + P_5$. Artin-Schreier reduction shows that the conductor of this cover is $2P_0 + 2P_1 + 2P_3$. This determines $N_2(10) = 13$, a result due to Serre.

We list here eight relations that can be used to obtain more good curves;

n	relation	n	relation
1	$[-3, -1, 2, 1, 1]$	5	$[-5, -1, 3, 2, 1]$
2	$[-1, -3, 1, 1, 2]$	6	$[-9, 1, 3, 2, 3]$
3	$[-7, 2, 3, 1, 1]$	7	$[-11, 1, 4, 3, 3]$
4	$[-3, 2, 1, -1, 1]$	8	$[-13, 2, 4, 3, 4]$

and reap the fruits by associating to a tuple $f_{i_1}, \ldots, f_{i_r}$ the corresponding fibre product $C_{f_{i_1}} \times_C C_{f_2} \times \cdots \times_C C_{f_{i_r}}$.

space	g	$\#C(\mathbb{F}_2$	interval	space	g	$\#$	interval
$\langle f_1 \rangle$	4	8	[8]	$\langle f_1, f_3, f_5 \rangle$	29	25	$[25-27]$
$\langle f_3 \rangle$	5	9	[9]	$\langle f_1, f_3, f_4 \rangle$	30	25	$[25-27]$
$\langle f_1, f_4 \rangle$	10	13	[13]	$\langle f_1, f_2, f_3 \rangle$	32	27	$[26-29]$
$\langle f_1, f_2 \rangle$	11	14	[14]	$\langle f_2, f_3, f_5 \rangle$	34	27	$[27-30]$
$\langle f_1, f_3 \rangle$	13	15	[15]	$\langle f_1, f_3, f_6 \rangle$	35	29	$[29-31]$
$\langle f_3, f_5 \rangle$	14	15	$[15-16]$	$\langle f_3, f_6, f_7 \rangle$	39	33	[33]
$\langle f_3, f_6 \rangle$	15	17	[17]	$\langle f_3, f_6, f_8 \rangle$	43	33	$[33-36]$
$\langle f_1, f_2, f_5 \rangle$	28	25	$[25-26]$	$\langle f_3, f_7, f_8 \rangle$	45	33	$[33-37]$

This reproduces many records of the tables and gives one improvement of the tables for curves over $\mathbb{F}_2$, namely for $g = 32$.

2.2. **Base of Genus 2 over $\mathbb{F}_2$.** Let C be the curve of genus 2 defined by the affine equation $y^2 + y = (x^2 + x)/(x^3 + x^2 + 1)$ over $\mathbb{F}_2$. It has six rational points

P_i $(i = 1, \ldots, 6)$ and class group $\mathrm{Jac}(C)(\mathbb{F}_2) \cong \mathbb{Z}/19\mathbb{Z}$. Let $\phi : C \to \mathrm{Jac}(C)$ be the Abel-Jacobi map given by $Q \mapsto Q - \deg(Q)\, P_1$. For a suitable numbering of the P_i the images of the P_i in $\mathrm{Jac}(C)(\mathbb{F}_2) \cong \mathbb{Z}/19\mathbb{Z}$ are as follows

i	1	2	3	4	5	6
$\phi(P_i)$	0	1	14	6	16	4

From this table we see that we have the following linear equivalence

$$-3P_1 - 3P_2 + 2P_3 + 2P_4 + P_5 + P_6 \sim 0. \qquad (2)$$

If f is the function with the left hand side of (2) as divisor the curve C_f obtained as the double cover of C defined by $w^2 + w = f$ has genus $g = 7$ and 10 rational points. This is optimal. Similarly the relation

$$-9P_1 + 2P_2 + 2P_3 + P_4 + 2P_5 + 2P_6 \sim 0$$

gives rise to a curve with $g = 8$ and 11 rational points, again an optimal curve over $\mathbb{F}_2$. Let Q be the divisor of zeros of the polynomial $x^3 + x^2 + 1$ on C. Then $\phi(Q) = 3$ and $-Q + \sum_{i=1}^{6} P_i \sim 0$ which gives us a curve of genus 9 with 12 points, again an optimal curve.

We list a number of divisors of functions f_n $(n = 1, \ldots, 10)$:

n	relation	n	relation
1	$[-3, -3, 2, 2, 1, 1]$	6	$[-11, 3, 2, 2, 3, 1]$
2	$[-9, 2, 2, 1, 2, 2]$	7	$[1, 0, -5, 2, 1, 1]$
3	$[1, 1, 1, 1, 1, 1] \sim Q$	8	$[3, -3, -3, 1, 1, 1]$
4	$[-1, -5, 1, 2, 2, 1]$	9	$[-7, 0, 2, 2, 2, 1]$
5	$[-5, -1, 2, 1, 1, 2]$	10	$[-13, 3, 1, 3, 3, 3]$

and we find as results:

space	g	$\#C(\mathbb{F}_2)$	interval	space	g	$\#C(\mathbb{F}_2)$	interval
$\langle f_1 \rangle$	7	10	[10]	$\langle f_2, f_5 \rangle$	20	19	$[19 - 21]$
$\langle f_2 \rangle$	8	11	[11]	$\langle f_2, f_6 \rangle$	22	21	$[21 - 22]$
$\langle f_3 \rangle$	9	12	[12]	$\langle f_2, f_{10} \rangle$	24	21	$[21 - 23]$
$\langle f_4, f_8 \rangle$	17	17	$[17 - 18]$	$\langle f_1, f_4, f_5 \rangle$	43	34	$[33 - 36]$
$\langle f_1, f_4 \rangle$	18	18	$[18 - 19]$	$\langle f_2, f_5, f_9 \rangle$	44	33	$[33 - 37]$

Again, we find one improvement, namely for $g = 43$.

We can also employ the class group for making an unramified cover of C of degree 19 in which P_1 splits completely. The genus of this cover is $g = 20$ and it contains 19 rational points. The interval is $[19 - 21]$.

2.3. **Base of Genus 3 over $\mathbb{F}_2$.** Consider now the curve C of genus 3 defined over $\mathbb{F}_2$ by the homogeneous equation

$$x^3y + x^3z + x^2y^2 + xz^3 + y^3z + y^2z^2 = 0.$$

It has 7 rational points and its class group $\mathrm{Jac}(C)(\mathbb{F}_2)$ is isomorphic to $\mathbb{Z}/71\mathbb{Z}$. We can number the rational points P_i $(i = 1, \ldots, 7)$ such that their images $\phi(P_i) =$ class of $(P_i - P_1)$ under the Abel-Jacobi map are as in the following table.

i	1	2	3	4	5	6	7
$\phi(P_i)$	0	1	34	55	10	14	49

There is a divisor Q_3 of degree 3 with $\phi(Q_3 - 3\,P_0) = 9$ and one of degree 5, say Q_5, with $\phi(Q_5 - 5\,P_0) = 35 \pmod{71}$; there is also a divisor Q_7 of degree 7 with $\phi(Q_7 - 7P_0) = 21$. We list a number of relations:

n	divisor	n	divisor
1	$[-3, 2, 1, 1, 2, 1, -4]$	5	$[-1, 3, 1, 1, 1, 2, -7]$
2	$[-11, 1, 2, 1, 2, 3, 2]$	6	$[2, -1, 1, 2, 2, 2, 1] \sim 3Q_3$
3	$[-13, 2, 1, 2, 2, 3, 3]$	7	$[1, 1, 1, 1, 1, 1, 1] \sim Q_7$
4	$[1, -5, 1, 2, 2, 1, 1] \sim Q_3$	8	$[-15, 4, 1, 3, 4, 1, 2]$

that yield the following results:

curve	g	$\#C(\mathbb{F}_2)$	interval	curve	g	$\#C(\mathbb{F}_2)$	interval
$\langle f_1 \rangle$	9	12	[12]	$\langle f_2, f_3 \rangle$	29	25	$[25 - 27]$
$\langle f_7 \rangle$	12	14	$[14 - 15]$	$\langle f_4, f_6 \rangle$	31	25	$[27 - 28]$
$\langle f_1, f_5 \rangle$	24	22	$[21 - 23]$	$\langle f_2, f_3, f_8 \rangle$	69	49	$[49 - 52]$

Note that for f_1 we have to do Artin-Schreier reduction. If the conductor would be $3P_0$ (resp. $3P_0 + P_6$) then C_{f_1} would give $g = 7, N = 11$ (resp. $g = 8, 12$), and these are impossible. So the conductor is $3P_0 + 3P_6$ giving ($g = 9, N = 12$), an optimal curve. Again we find one improvement (for $g = 24$) for our tables. The interval for genus 69 comes from page 121 of [6].

2.4. **Base of Genus** 5 **over** $\mathbb{F}_2$**.** Consider the fibre product C of C_1 given by $y^2 + y = x^3 + x$ and C_2 given by $y^2 + y = x^5 + x^3$ over the x-line. This is an optimal curve of genus 5 with 9 points. Its Jacobian is isogenous with the product of the three Jacobians of the curves C_1 and C_2 and the curve C_2' defined by $y^2 + y = x^5 + x$. The corresponding L-polynomials are $2t^2 + 2t + 1$, $4t^4 + 4t^3 + 2t^2 + 2t + 1$ and $4t^4 + 4t^3 + 4t^2 + 2t + 1$. The class groups are $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/13\mathbb{Z}$ and $\mathbb{Z}/15\mathbb{Z}$. The nine rational points are P_∞ and eight points that can be identified by their (x, y_1, y_2, y_3)-coordinates. Writing the class group of C as $\mathbb{Z}/65\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ we have the following table:

i	1	2	3	4	5	6	7	8	9
$\phi(P_i)$	$(0,0)$	$(1,1)$	$(51,14)$	$(64,1)$	$(14,14)$	$(57,11)$	$(47,4)$	$(8,11)$	$(18,4)$

The relation $[1, 2, -13, 1, 2, 2, 2, 2, 1]$ gives a curve with $g = 16$ and 17 points (interval $[17 - 18]$). The relation $[1, 1, -11, 1, -1, 3, 2, 1, 3]$ gives a curve with $g = 16$ and 16 points. The fibre product has genus $g = 39$ with 31 points. The combination of the relations $[1, 2, -13, 1, 2, 2, 2, 2, 1]$ and $[3, 3, -17, 1, 2, 1, 3, 1, 3]$ gives a curve with $g = 42$ and 33 points (interval $[33 - 35]$).

2.5. **Genus** 1 **over** $\mathbb{F}_3$**.** Consider the elliptic curve defined by the equation $y^2 = x^3 + 2x + 1$ over $\mathbb{F}_3$. It has 7 points P_i with $i = 0, \ldots, 7$ such that $P_i \mapsto i \pmod 7$ defines an isomorphism $C(\mathbb{F}_3) \cong \mathbb{Z}/7\mathbb{Z}$. The relations $[-4, -1, 0, 1, 2, 1, 1]$ and $[-4, -2, 1, 2, 1, 1, 1]$ give a fibre product of genus $g = 30$ with 38 points that improves the interval $[37 - 46]$ slightly.

2.6. **Curves of Genus** 2 **over** $\mathbb{F}_3$**.** The curve C of genus 2 over $\mathbb{F}_3$ given by $y^2 = x^5 + x^3 + x + 1$ has zeta function $(9t^4 + 9t^3 + 5t^2 + 3t + 1)/(1 - t)(1 - 3t)$. The curve has 7 rational points and class group $\mathbb{Z}/27\mathbb{Z}$. The 7 rational points map to $0, 1, 26, 17, 10, 23, 4$ in the group. We have the relations $[1, 1, 1, 1, -2, 2, -4]$ and

$[1,1,1,2,-1,1,-5]$ and the corresponding fibre product has $g = 44$ with 47 points. This gives a new entry for the table, albeit not a very strong one (resulting interval $[47-61]$).

2.7. Base Curve of Genus 3 over $\mathbb{F}_3$. The curve C of genus 3 given by

$$2\,x^4 + x^3z + 2\,x^2y^2 + x^2yz + x^2z^2 + 2\,xz^3 + 2\,y^4 + 2\,y^3z + 2y^2z^2 = 0$$

has 10 rational points and class group $\mathbb{Z}/204$. The 10 points map to

$$0, 72, 129, 59, 182, 121, 172, 45, 47, 26$$

in $\mathbb{Z}/204$. We have the relation $[1,2,1,1,1,1,1,1,-10,1]$ leading to an Artin-Schreier cover with genus 18 with 28 points improving the interval $[26-31]$ to $[28-31]$.

3. Subgroups of the Class group

Let C be a smooth complete irreducible curve over $\mathbb{F}_q$ with Jacobian $\mathrm{Jac}(C)$ and class group $G = \mathrm{Jac}(C)(\mathbb{F}_q)$. We choose a rational point P, provided there is one, and consider the morphism $\phi : C \to \mathrm{Jac}(C)$ given by $Q \mapsto Q - \deg(Q)P$. If H is a subgroup of G of index d containing the images $\{\phi(P_i) : i \in I\}$ then there exists an unramified degree d cover $\tilde{C}$ of C in which the points P_i with $i \in I$ split completely. Choosing C and I appropriately can produce curves with many points.

3.1. Base of Genus 4 over $\mathbb{F}_2$. The reader might think that it is necessary to start with a curve with many points. To dispel this idea start with the genus 4 curve given by the equation

$$y^2 + y = (x^7 + x^5 + 1)/(x^2 + x)$$

over $\mathbb{F}_2$ which has three rational points P_1, P_2, P_3. The L-polynomial is $16\,t^8 + 4\,t^6 + 4\,t^5 + 4\,t^4 + 2\,t^3 + t^2 + 1$, hence the class number is 32. In fact, the class group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$ and the differences $P_i - P_1$ map to $(0,0)$, $(1,0)$ and $(1,8)$. Hence there exists an étale cover of degree 8 in which the three points split completely. This gives a curve of genus 25 with 24 points. This is optimal.

Or start with the hyperelliptic curve of genus 5 given by

$$y^2 + y = (x^9 + x^7 + x^3 + x + 1)/(x^2 + x)$$

again with three rational points and class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$ with the points now mapping to $(0,0)$, $(1,0)$ and $(0,15)$. This gives an étale cover of degree 15 with genus $g = 61$ with 45 points, very close to the best upper bound 47 and improving the interval $[41-47]$ of [6], p. 121.

3.2. Genus 2 over $\mathbb{F}_3$. We consider the curve C of genus 2 over $\mathbb{F}_3$ given by the affine equation $y^3 - y = x - 1/x$. The zeta function is $(9t^4 + 12t^3 + 10t^2 + 4t + 1)/(1-t)(1-3t)$ and its class group is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The images of the 8 rational points under the Abel-Jacobi map are

$$[0,0], [2,4], [1,0], [1,4], [0,1], [2,3], [5,5], [3,5]$$

in $(\mathbb{Z}/6\mathbb{Z})^2$. The subgroup of the class group of index 12 given by $x_1 + x_2 \equiv 0 \pmod 3$ and $x_1 \equiv x_2 \equiv 0 \pmod 2$ contains $[0,0]$ and $[2,4]$ and thus leads to a cover of degree 12 of genus $g = 13$ and with 24 rational points (interval $[24-25]$). There is a place Q_2 of degree 2 which maps to $[2,1]$ in the class group. The relation

$\sum_{i=1}^{8} P_i \sim 4Q_2$ leads to an Artin-Schreier cover with $g = 14$ with 24 points (interval $[24-25]$).

An alternative case: consider $y^2 = x(x^2+1)(x^2-x-1)$ over $\mathbb{F}_3$ of genus 2 with 6 points and class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$. The points map under an Abel-Jacobi map to

$$[0,0], [1,0], [0,9], [0,1], [1,7], [1,3].$$

So the index 10 subgroup defined by $x_2 = 0 \pmod{10}$ gives a cover of degree 10 of genus 11 with 20 points. (Interval $[20-22]$.)

Finally, consider the curve C defined over $\mathbb{F}_3$ by the equation $y^2 = 2x^5 + x^4 + x$. It has 6 rational points and zeta function $(9t^4 + 6t^3 + 4t^2 + 2t + 1)/(1-t)(1-3t)$ and class group $\mathbb{Z}/22\mathbb{Z}$. The six points map to $0, 11, 12, 10, 3, 19 \pmod{22}$. We thus see that there is an étale cover of degree 11 with genus 12 and 22 rational points (interval $[22-23]$).

Let C be the curve of genus 2 over $\mathbb{F}_3$ defined by the equation $y^3 - y = x + 1/x$. The zeta function of this curve is

$$(81t^4 + 90t^3 + 43t^2 + 10t + 1)/(1-t)(1-9t)$$

and its class group is $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$. The curve possesses 20 rational points over $\mathbb{F}_9$. The images of these points under a suitable Abel-Jacobi map are given by

$$[0,13], [0,8], [7,9], [8,12], [14,13], [1,8], [6,12], [9,9], [14,2], [1,4],$$
$$[2,9], [13,12], [4,7], [11,14], [9,6], [6,0], [4,8], [11,13], [0,6], [0,0].$$

The equation $5a + 5b = 0 \pmod{15}$ defines a subgroup of index 3 in $\mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ containing the images of 10 points, hence we get a curve $\tilde{C}$, an unramified cover of C of degree 3 with genus 4 with 30 points, the maximum possible.

Similarly, the six points P_i corresponding to pairs $[a,b]$ with $a \equiv b \equiv 0 \pmod 3$ lie in a subgroup of index 9. So there is an unramified cover $\tilde{C}$ of degree 9 and genus 10 with $9 \times 6 = 54$ rational points. The interval in the tables is $[54-55]$.

3.3. **Base Curve of Genus 3 over $\mathbb{F}_3$.** Consider the plane curve of degree 4 over $\mathbb{F}_3$ given by

$$2\,x^3y + 2\,x^3z + x^2y^2 + xz^3 + 2\,y^3z + yz^3 = 0.$$

This curve has 10 points over $\mathbb{F}_3$ and has class group isomorphic to $(\mathbb{Z}/14\mathbb{Z})^2$ and the 10 points go to

$$[0,0], [1,0], [6,13], [7,3], [13,7], [4,12], [11,11], [6,2], [4,7]]$$

The subgroup defined by $x_2 \equiv 0 \pmod 7$ has index 7 and contains the images of 4 points. The corresponding curve over $\mathbb{F}_3$ has genus 15 with 28 rational points; this is optimal.

Or consider the hyperelliptic curve given by the equation

$$y^2 + (x^3 - x)y = x^7 - x^2 + x$$

over $\mathbb{F}_3$. It is of genus 3, has five rational points mapping in the class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ to $(0,0,0,), (0,0,6), (0,1,0), (0,0,11), (0,0,1)$. The index 12 subgroup containing the first three points gives rise to a cover of genus $g = 25$ with 36 points (the interval being $[36-40]$).

The hyperelliptic curve $y^2 = x^7 - x^2 + x$ over $\mathbb{F}_3$ has similarly a class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$ and an index 14 subgroup containing three points giving rise to a curve of genus 29 with 42 points (interval $[42-44]$).

The hyperelliptic curve of genus 4 over $\mathbb{F}_3$ given by $y^2 + xy = x^9 - x$ has class group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/126\mathbb{Z}$ and the six rational points map to

$$(0,0),(1,0),(0,125),(0,1),(0,14),(0,112)$$

and so the index 14 subgroup containing four of these points gives rise to a cover of genus 43 with 56 points improving the interval $[55-60]$.

3.4. **Examples over $\mathbb{F}_4$.** Let C be the hyperelliptic curve of genus 2 over $\mathbb{F}_4$ given by $y^2 + y = x^5 + x^3 + x$ with class group $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$. It has a cover of degree 7 in which three points split completely giving a curve of genus 8 with 21 rational points (interval $[21-24]$).

3.5. **Genus 3 over $\mathbb{F}_9$.** Let α be a generator of the multiplicative group $\mathbb{F}_9^*$ and consider the curve C of genus 3 given by $y^3 - y = \alpha^2 x^4$ over the field $\mathbb{F}_9$. This curve has 28 rational points, the bitangent points of a plane model. The class group is of the form $(\mathbb{Z}/4\mathbb{Z})^6$. The 28 points map under an Abel-Jacobi map to the following elements in $(\mathbb{Z}/4\mathbb{Z})^6$.

[0, 0, 0, 0, 0, 0], [3, 1, 3, 3, 0, 1], [3, 2, 1, 0, 3, 0], [2, 0, 3, 1, 1, 3], [2, 1, 1, 1, 1, 3],
[2, 2, 0, 1, 1, 3], [3, 1, 0, 1, 1, 3], [1, 1, 0, 0, 1, 3], [2, 1, 0, 2, 1, 3], [1, 3, 3, 2, 0, 3],
[1, 2, 2, 3, 2, 0], [0, 2, 3, 2, 1, 2], [2, 0, 0, 0, 1, 2], [1, 1, 3, 1, 2, 0], [3, 2, 1, 2, 0, 3],
[0, 0, 1, 3, 2, 1], [0, 3, 1, 1, 2, 2], [2, 0, 2, 3, 3, 2], [3, 3, 0, 3, 2, 3], [3, 1, 2, 0, 1, 0],
[0, 3, 2, 0, 0, 2], [1, 1, 1, 2, 1, 1], [1, 2, 0, 3, 3, 3], [0, 0, 3, 2, 3, 1], [2, 1, 0, 1, 1, 0],
[2, 1, 0, 1, 0, 2], [2, 1, 0, 1, 2, 3], [2, 1, 0, 1, 1, 3]

The index 2 subgroup of the class group defined by the equation $2x_1 \equiv 0 \pmod 4$ contains the images of 16 points. The corresponding étale covering of C of degree 2 is of genus 5 and has 32 points (interval $[32-35]$).

The index 4 subgroup of the class group defined by the equation $x_2+x_3+x_4+x_5 \equiv 0 \pmod 4$ contains the images of 12 points. We thus find a degree 4 étale cover of C of genus 9 with 48 points (interval $[48-50]$).

4. Using the class group over extension fields

Let C a smooth projective curve defined over $\mathbb{F}_q$ of genus g and with m rational points. Let J be its Jacobian variety. It is easy to see that if $Z(C,t)$ is the zeta function of $C/\mathbb{F}_q$ and $Z_n(C,t)$ the zeta function of C considered as a curve over $\mathbb{F}_{q^n}$ then we have $Z_n(C,t^n) = \prod_\zeta Z(C,\zeta t)$, where the product is taken over the nth roots ζ of 1. If we write $Z(C,t) = L(C,t)/(1-t)(1-qt)$ and $Z_n(C,t) = L_n(C,t)/(1-t)(1-q^n t)$ we get $L_n(C,1) = \prod_\zeta L(C,\zeta)$. Moreover, we know that $\#J(\mathbb{F}_q) = L(C,1)$ and $\#J(\mathbb{F}_{q^n}) = L_n(C,1)$ and we thus get

$$[J(\mathbb{F}_{q^n}) : J(\mathbb{F}_q)] = \prod_{\zeta^n=1,\zeta\neq 1} L(C,\zeta).$$

Since under the Abel-Jacobi map the $\mathbb{F}_q$-rational points of C map to $J(\mathbb{F}_q)$ we conclude that there exists an unramified cover of C defined over $\mathbb{F}_{q^n}$ of degree $d = \prod_{\zeta^n=1,\zeta\neq 1} L(C,\zeta)$ in which all the $\mathbb{F}_q$-rational points split completely. Thus we find a curve $\tilde{C}$ of genus $d(g-1)+1$ with at least dm rational points. This idea was exploited very succesfully by Niederreiter and Xing in their papers [2] up to [6]. Here we employ it more systematically by going through all isomorphism classes of curves of low genera. This improves the tables at many places.

We now list all possible L-functions of genus 2 curves over $\mathbb{F}_2$. By N_i we mean $\#C(\mathbb{F}_{2^i})$.

f	$[N_1, N_2]$	Cl	L
$(x^2+x)/(x^3+x^2+1)$	$[6,6]$	$\mathbb{Z}/19\mathbb{Z}$	$4t^4+6t^3+5t^2+3t+1$
$(x^3+x+1)/(x^3+x^2+1)$	$[0,6]$	(0)	$4t^4-6t^3+5t^2-3t+1$
$1/(x^3+x^2+1)$	$[2,6]$	$\mathbb{Z}/3\mathbb{Z}$	$4t^4-2t^3+t^2-t+1$
$x/(x^3+x^2+1)$	$[4,6]$	$\mathbb{Z}/9\mathbb{Z}$	$4t^4+2t^3+t^2+t+1$
$x^2/(x^3+x^2+1)$	$[4,10]$	$\mathbb{Z}/11\mathbb{Z}$	$4t^4+2t^3+3t^2+t+1$
$(x^3+1)/(x^3+x^2+1)$	$[2,10]$	$\mathbb{Z}/5\mathbb{Z}$	$4t^4-2t^3+3t^2+t+1$
$1/x(x^2+x+1)$	$[3,7]$	$\mathbb{Z}/6\mathbb{Z}$	$4t^4+t^2+1$
$(x+1)/x(x^2+x+1)$	$[5,7]$	$\mathbb{Z}/14\mathbb{Z}$	$4t^4+4t^3+3t^2+2t+1$
$(x^3+x^2+1)/(x(x^2+x+1)$	$[1,7]$	$\mathbb{Z}/2\mathbb{Z}$	$4t^4-4t^3+3t^2-2t+1$
$(x^3+x^2+1)/x(x+1)$	$[3,3]$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$4t^4-t^2+1$
$1/x+x^3$	$[4,4]$	$\mathbb{Z}/8\mathbb{Z}$	$4t^4+2t^3+t+1$
$1/x+x^2+x^3$	$[2,8]$	$\mathbb{Z}/4\mathbb{Z}$	$4t^4-2t^3+2t^2-t+1$
$1/x+1+x^3$	$[2,4]$	$\mathbb{Z}/2\mathbb{Z}$	t^4-2t^3-t+1
$1/x+1+x^2+x^3$	$[4,8]$	$\mathbb{Z}/10\mathbb{Z}$	$4t^4+2t^3+2t^2+t+1$
x^5	$[3,5]$	$\mathbb{Z}/5\mathbb{Z}$	$4t^4+1$
x^5+x^3+x	$[3,9]$	$\mathbb{Z}/7\mathbb{Z}$	$4t^4+2t^2+1$
x^5+x	$[5,9]$	$\mathbb{Z}/15\mathbb{Z}$	$4t^4+4t^3+4t^2+2t+1$
x^5+x+1	$[1,5]$	$\mathbb{Z}/3\mathbb{Z}$	$4t^4-4t^3+4t^2-2t+1$
x^5+x^3	$[5,5]$	$\mathbb{Z}/13\mathbb{Z}$	$4t^4+4t^3+2t^2+2t+1$
x^5+x^3+1	$[1,9]$	(0)	$4t^4-4t^3+2t^2-2t+1$

If we now take $n=2$ we get curves defined over $\mathbb{F}_4$; for example we find the following cases that do not give improvements of the table for $\mathbb{F}_4$ but good realizations of the present records.

$[N_1, N_2]$	g	$\#C(\mathbb{F}_4)$	interval
$[5,9]$	4	15	[15]
$[4,10]$	6	20	[20]
$[3,9]$	8	21	$[21-24]$

With $n=3$ we get the following cases

$[N_1, N_2]$	g	$\#C(\mathbb{F}_8)$	interval
$[5,7]$	8	35	$[35-42]$
$[5,5]$	14	65	[65]
$[4,4]$	20	76	$[68-83]$

Here the case $g=14$ with $\#C(\mathbb{F}_4)=65$ is optimal; the case $g=8$ with 35 rational points equals the lower bound of the interval $[35-42]$ of the tables while $g=20$ with 76 points improves the interval $[68-83]$ of the tables.

Starting with the curves of genus 2 over $\mathbb{F}_4$ gives the following results:

$[N_1, N_2]$	g	$\#C(\mathbb{F}_{16})$	interval
$[9,24]$	9	72	$[72-81]$
$[9,25]$	10	81	$[81-87]$
$[8,24]$	11	80	$[80-91]$
$[8,26]$	12	88	$[83-97]$
$[7,27]$	15	98	$[98-113]$
$[7,31]$	17	112	$[112-123]$

And the same can be done starting with the curves of genus 2 over $\mathbb{F}_8$. Here there are still many empty places in the tables due to the lack of good examples.

$[N_1, N_2]$	g	$\#C(\mathbb{F}_{64})$	interval
$[18, 54]$	20	342	
$[17, 63]$	25	408	
$[17, 65]$	26	425	
$[16, 64]$	27	416	
$[16, 70]$	30	464	
$[15, 71]$	33	480	
$[14, 66]$	34	462	
$[15, 75]$	35	510	
$[14, 70]$	36	490	
$[15, 79]$	37	540	
$[14, 74]$	38	518	
$[13, 67]$	39	494	$[489 - 650]$
$[14, 78]$	40	546	
$[14, 80]$	41	560	
$[14, 82]$	42	574	
$[12, 66]$	44	516	
$[13, 79]$	45	572	
$[13, 81]$	46	585	
$[13, 83]$	47	598	
$[12, 74]$	48	564	
$[13, 87]$	49	624	
$[12, 78]$	50	588	

We can use curves of higher genus too. Going through all non-hyperelliptic curves of genus 3 over $\mathbb{F}_2$ gives two improvements for the table over $\mathbb{F}_4$:

$[N_1, N_2, N_3]$	g	$\#C(\mathbb{F}_4)$	interval
$[5, 9, 5]$	15	35	$[33 - 37]$
$[5, 11, 5]$	17	40	$[40]$
$[4, 12, 7]$	27	52	$[50 - 56]$

and considering these curves over $\mathbb{F}_4$ gives three improvements for the table over $\mathbb{F}_{16}$:

$[N_1, N_2, N_3]$	g	$\#C(\mathbb{F}_{16})$	interval
$[4, 12, 7]$	27	156	$[145 - 176]$
$[3, 11, 6]$	35	187	
$[5, 11, 5]$	41	220	$[216 - 249]$

Going through the hyperelliptic curves of genus 4 over $\mathbb{F}_2$ yields one improvement of the table over $\mathbb{F}_4$: the curve with $[N_1, N_2, N_3, N_4] = [6, 10, 6, 26]$ yields a curve of genus 28 with 54 points over $\mathbb{F}_4$; the interval was $[53 - 58]$.

Taking the curve of genus 4 over $\mathbb{F}_2$ with $[N_1, N_2, N_3, N_4] = [8, 8, 8, 16]$ over $\mathbb{F}_2$ and L-polynomial $16t^8 + 40t^7 + 56t^6 + 56t^5 + 44t^4 + 28t^3 + 14t^2 + 5t + 1$ gives a degree 13 cover with genus 40 and 104 points, improving the interval $[103 - 141]$.

Similarly, we can list all pairs $[N_1, N_2]$ with $N_i = \#C(\mathbb{F}_{3^i})$ occuring for curves of genus 2 over $\mathbb{F}_3$. This yields the following harvest.

$[N_1, N_2]$	g	$\#C(\mathbb{F}_9)$	interval
$[8, 14]$	5	32	$[32 - 35]$
$[7, 15]$	6	35	$[35 - 40]$
$[6, 16]$	8	42	$[40 - 47]$
$[6, 18]$	9	48	$[48 - 50]$
$[6, 20]$	10	54	$[54]$

Doing the same for curves of genus 2 over $\mathbb{F}_9$ gives the following.

$[N_1, N_2]$	g	$\#C(\mathbb{F}_{81})$	interval
$[20-68]$	26	500	
$[19-75]$	30	551	
$[18-78]$	33	576	
$[18-80]$	34	594	$[494-689]$
$[18-82]$	35	612	
$[18-86]$	37	648	$[568-742]$
$[17-83]$	38	629	
$[17-85]$	39	646	
$[17-87]$	40	663	
$[17-89]$	41	680	
$[17-91]$	42	697	
$[16-86]$	43	672	
$[16-90]$	45	704	
$[16-92]$	46	720	
$[15-85]$	47	690	
$[16-96]$	48	752	$[676-885]$
$[16-86]$	49	672	$[656-898]$
$[16-100]$	50	784	

Similarly, the genus 2 curves over $\mathbb{F}_3$ give with $n = 3$:

$[N_1, N_2]$	g	$\#C(\mathbb{F}_{27})$	interval
$[8, 10]$	26	200	
$[7, 13]$	29	196	
$[7, 11]$	40	273	$[244-346]$

For the case of non-hyperelliptic curves of genus 3 over $\mathbb{F}_3$ we find the following cases:

m	$\#C(\mathbb{F}_3)$	$L(-1)$	g	$\#\tilde{C}(\mathbb{F}_9)$	interval
687439	8	12	25	96	$[82-108]$
787452	8	13	27	104	$[91-114]$
687411	7	17	35	119	$[116-139]$
787567	7	18	37	126	$[120-145]$
884286	7	20	41	140	$[128-158]$
687541	7	21	43	147	$[120-164]$

Here we use the following notation for plane curves of degree 3 over $\mathbb{F}_3$. We write the polynomials in x, y, z in pure lexicographic order and use then the coefficients c_i $(i = 1, \ldots, 15)$ to associate the natural number $m = \sum_{i=1}^{15} c_i 3^{i-1}$ to the curve. So x^4 corresponds to 1 and the Klein curve $x^3y + y^3z + z^3x = 0$ to 196833.

5. Tables

Table p=2.

$g\backslash q$	2	4	8	16	32	64	128
1	5	9	14	25	44	81	150
2	6	10	18	33	53	97	172
3	7	14	24	38	64	113	192
4	8	15	25	45	71–74	129	215
5	9	17	29–30	49–53	83–85	132–145	227–234
6	10	20	33–35	65	86–96	161	243–258
7	10	21–22	34–38	63–69	98–107	177	262–283
8	11	21–24	35–42	62–75	97–118	169–193	276–302
9	12	26	45	72–81	108–128	209	288–322
10	13	27	42–49	81–87	113–139	225	296–345
11	14	26–29	48–53	80–91	120–150	201–236	294–366
12	14–15	29–31	49–57	88–97	129–161	257	321–388
13	15	33	56–61	97–102	129–172	225–268	
14	15–16	32–35	65	97–107	146–183	241–284	353–437
15	17	35–37	57–67	98–113	158–194	258–300	386–455
16	17–18	36–38	56–71	95–118	147–204	267–316	
17	17–18	40	63–74	112–123	154–212		
18	18–19	41–42	65–77	113–129	161–220	281–348	
19	20	37–43	60–80	129–134	172–228	315–364	
20	19–21	40–45	76–83	127–139	177–236	342–380	
21	21	44–47	72–86	129–145	185–243	281–396	
22	21–22	42–48	74–89	129–150		321–412	
23	22–23	45–50	68–92	126–155			
24	22–23	49–52	81–95	129–161	225–267	337–444	513–653
25	24	51–53	86–97	144–165		408–460	
26	24–25	55	82–100	150–171		425–476	
27	24–25	52–56	96–103	156–176	213–290	416–492	
28	25–26	54–58	97–106	145–181	257–298	513	577–745
29	25–27	52–60	97–109	161–186	227–305		
30	25–27	53–61	96–112	162–191	273–313	464–535	609–784
31	27–28	60–63	89–115	165–196		450–547	578–807
32	27–29	57–65	90–118				
33	28–29	65–66	97-121	193–207		480–570	
34	27–30	65–68	98–124	183–212		462–581	
35	29–31	64–69	112–127	187–217	253–351	510–593	
36	30–31	64–71	112–130	185–222		490–604	705–917
37	30–32	66–72	121–132	208–227		540–616	
38	30–33	64–74	129–135	193–233	291–375	518–627	
39	33	65–75	120–138	194–238		494–638	
40	32–34	75–77	103–141	225–243	293–390	546–649	
41	33–35	65–78	118–144	220–249	308–398	560–661	
42	33–35	75–80	129–147	209–254	307–405	574–672	
43	34–36	72–81	116–150	226–259	306–413	546–684	
44	33–37	68–83	130–153	226–264	325–420	516–695	
45	33–37	80–84	144–156	242–268	313–428	572–706	
46	34–38	81–86	129–158	243–273		585–717	
47	36–38	73–87	126–161			598–729	
48	34–39	80–89	128–164	243–282		564–740	
49	36–40	81–90	130–167	213–286		624–751	913–1207
50	40	91–92	130–170	255–291		588–762	

Table p=3.

$g\backslash q$	3	9	27	81
1	7	16	38	100
2	8	20	48	118
3	10	28	56	136
4	12	30	64	154
5	13	32–35	72–75	160–172
6	14	35–40	76–85	190
7	16	40–43	82–95	180–208
8	17–18	42–47	92–105	226
9	19	48–50	99–113	244
10	20–21	54	94–123	226–262
11	20–22	55–58	100–133	220–280
12	22–23	56–62	109–143	298
13	24–25	64–65	136–153	256–312
14	24–26	56–69		278–330
15	28	64–73	136–170	292–348
16	27–29	74–77	144–178	370
17	25–30	74–81	128–185	288–384
18	28–31	67–84	148–192	306–401
19	32	84–88	145–199	
20	30–34	70–91		
21	32–35	88–95	163–213	352–455
22	30–36	78–98		
23	32–37	92–101		
24	31–38	91–104	208–234	
25	36–40	96–108	196–241	392–527
26	36–41	110–111	200–248	500–545
27	39–42	104–114		
28	37–43	105–117		
29	42–44	104–120	196–269	
30	38–46	91–123	196–276	551–617
31	40–47	120–127		460–635
32	40–48	92–130		
33	46–49	128–133	220–297	576–671
34	46–50	111–136		594–689
35	47–51	119–139		612–707
36	48–52	118–142	244–318	730
37	52–54	126–145	236–325	648–742
38		105–149		629–755
39	48–56	140–152	271–340	646–768
40	56–57	118–155	273–346	663–781
41	50–58	140–158		680–795
42	52–59	122–161	280–360	697–808
43	56–60	147–164		672–821
44	47–61	119–167	278–374	
45	54–62	136–170		704–847
46	55–63	162–173		720–859
47	54–65	154–177	299–395	690–872
48	55–66	163–180	325–402	752–885
49	64–67	168–183	316–409	768–898
50	63–68	182–186	312–416	784–911

For exhaustive references we refer to the bibliography of the tables, [10]. For general background on curves over finite fields we refer to the book by Stichtenoth [9] and for general background on curves over finite fields with many points to Serre's notes [8] and for an overview of the methods of Niederreiter and Xing to [6].

References

[1] E. Howe, K. Lauter: Improved upper bounds for the number of points on curves over finite fields. *Ann. Inst. Fourier* **53** (2003), p. 1677–1737. and: Corrigendum to: Improved upper bounds for the number of points on curves over finite fields. *Ann. Inst. Fourier (Grenoble)* **57** (2007), no. 3, pp. 1019–1021.

[2] H. Niederreiter, C. P. Xing: Explicit global function fields over the binary field with many rational places. *Acta Arithm.* **75** (1996), p. 383-396.

[3] H. Niederreiter, C. P. Xing: Cyclotomic function fields, Hilbert class fields and global function fields with many rational places. *Acta Arithm.* **79** (1997), p. 59-76.

[4] H. Niederreiter, C. P. Xing: Global function fields fields with many rational points over the ternary field. *Acta Arithm.* **83** (1998), p. 65–86.

[5] H. Niederreiter, C. P. Xing: Algebraic curves with many rational points over finite fields of characteristic 2. In: *Proc. Number Theory Conference* (Zakopane 1997), de Gruyter, 1999, Berlin, p. 359–380.

[6] H. Niederreiter, C.P. Xing: *Rational Points on Curves over Finite Fields—Theory and Applications*, London Math. Soc. Lecture Note Ser., vol. 285, Cambridge University Press, 2002.

[7] J.-P. Serre: Letter to G. van der Geer. September 1, 1997.

[8] J.-P. Serre: Rational points on curves over finite fields. Notes of lectures at Harvard University 1985.

[9] H. Stichtenoth: Algebraic Function Fields and Codes. Universitext. Springer Verlag 1993.

[10] G. van der Geer, M. van der Vlugt: Tables of curves with many points. *Math. Comp.* **69** (2000), no. 230, pp. 797–810. http://www.science.uva.nl/∼geer/tables-mathcomp20.pdf

Korteweg-de Vries Instituut, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

E-mail address: geer@science.uva.nl