COUNTING POINTS ON THE FRICKE-MACBEATH CURVE OVER FINITE FIELDS

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ABSTRACT. The Fricke-Macbeath curve is a smooth projective algebraic curve of genus 7 with automorphism group $\mathrm{PSL}_2(\mathbb{F}_8)$. We recall two models of it (introduced, respectively, by Maxim Hendriks and by Bradley Brock) defined over $\mathbb{Q}$, and we establish an explicit isomorphism defined over $\mathbb{Q}(\sqrt{-7})$ between these models. Moreover, we decompose upto isogeny over $\mathbb{Q}$ the jacobian of one of these models. As a consequence we obtain a simple formula for the number of points over $\mathbb{F}_q$ on (the reduction of) this model, in terms of the elliptic curve with equation $y^2 = x^3 + x^2 - 114x - 127$. Moreover, twists by elements of $\mathrm{PSL}_2(\mathbb{F}_8)$ of the curve over finite fields are described. The curve leads to a number of new records as maintained on `manypoints.org` of curves of genus 7 with many rational points over finite fields.

1. INTRODUCTION

It is well-known that an algebraic curve of genus $g > 1$ over $\mathbb{C}$ has at most $84(g-1)$ automorphisms. A curve attaining this bound is called a Hurwitz curve. The corresponding Riemann surface can in this case be described as $\Gamma\ \mathcal{H}$ in which Γ is a normal subgroup of finite index in the triangle group $G_{2,3,7}$, acting in the classical way on the complex upper half plane $\mathcal{H}$. See, e.g., §3.19 of Shimura's paper [Sh67] and §5.3 of the exposition by Elkies [El98] for details The plane curve with equation $x^3y + y^3z + z^3z = 0$, named after Felix Klein who studied it in 1879 in his paper [K79], is the unique example up to isomorphisms for genus $g = 3$. The next example occurs for $g = 7$ and was introduced as a Riemann surface by Robert Fricke in 1899 [F99]. Explicit equations realizing Fricke's example as an algebraic curve, were presented in 1965 by A.M. Macbeath [M65], see also W.L. Edge's paper [Ed67] which appeared two years later. Again, upto isomorphisms over $\mathbb{C}$ there is a unique curve of genus 7 admitting 504 automorphisms; here and elsewhere it is called the Fricke-Macbeath curve. Whereas Edge derives the equations first presented by Macbeath by starting from the property that they need to define a curve in $\mathbb{P}^6$ having a given subgroup of order 504 in $\mathrm{PGL}_7(\mathbb{C})$ as automorphism group, there is an alternative, very natural way to find the curve, as is explained in a letter dated 24-vii-1990 of J-P. Serre to S.S. Abhyankar [Se90]. Namely, Serre observes that $G = \mathrm{PSL}_2(\mathbb{F}_8)$ is a transitive subgroup of the alternating group A_9 (which in fact follows from the action of G on the 9 points in $\mathbb{P}^1(\mathbb{F}_8)$). The stabilizer $S \subset G$ of any of these 9 points then makes $X \to X/G$ the normal closure of $X/S \to X/G$, where we denote the desired curve as X. Both X/S and X/G are rational curves, and the ramification of the resulting degree 9 map $\mathbb{P}^1 \to \mathbb{P}^1$ is known and occurs only over three points. This information suffices to determine the degree 9 map explicitly, and hence to find the curve X.

The equations described by Macbeath (and explained in detail by Edge) define a curve $M \subset \mathbb{P}^6$, given (with ζ_7 a primitive 7th root of unity) by the ideal with generators

$$\begin{array}{l}
M:\\
x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2,\\
x_0^2+\zeta_7x_1^2+\zeta_7^2x_2^2+\zeta_7^3x_3^2+\zeta_7^4x_4^2+\zeta_7^5x_5^2+\zeta_7^6x_6^2,\\
x_0^2+\zeta_7^6x_1^2+\zeta_7^5x_2^2+\zeta_7^4x_3^2+\zeta_7^3x_4^2+\zeta_7^2x_5^2+\zeta_7x_6^2,\\
\left(\zeta_7^5-\zeta_7^2\right)x_1x_4+\left(\zeta_7^6-\zeta_7\right)x_3x_5+\left(-\zeta_7^4+\zeta_7^3\right)x_0x_6,\\
\left(-\zeta_7^4+\zeta_7^3\right)x_0x_1+\left(\zeta_7^5-\zeta_7^2\right)x_2x_5+\left(\zeta_7^6-\zeta_7\right)x_4x_6,\\
\left(-\zeta_7^4+\zeta_7^3\right)x_1x_2+\left(\zeta_7^6-\zeta_7\right)x_0x_5+\left(\zeta_7^5-\zeta_7^2\right)x_3x_6,\\
\left(-\zeta_7^4+\zeta_7^3\right)x_2x_3+\left(\zeta_7^5-\zeta_7^2\right)x_0x_4+\left(\zeta_7^6-\zeta_7\right)x_1x_6,\\
\left(\zeta_7^6-\zeta_7\right)x_0x_2+\left(-\zeta_7^4+\zeta_7^3\right)x_3x_4+\left(\zeta_7^5-\zeta_7^2\right)x_1x_5,\\
\left(\zeta_7^6-\zeta_7\right)x_1x_3+\left(-\zeta_7^4+\zeta_7^3\right)x_4x_5+\left(\zeta_7^5-\zeta_7^2\right)x_2x_6,\\
\left(\zeta_7^5-\zeta_7^2\right)x_0x_3+\left(\zeta_7^6-\zeta_7\right)x_2x_4+\left(-\zeta_7^4+\zeta_7^3\right)x_5x_6.
\end{array}$$

A consequence of a very general criterion of Girondo, Torres-Teigell, and Wolfart [GTW14] is that it is possible to define the Fricke-Macbeath curve as an algebraic curve over $\mathbb{Q}$. As part of his PhD research, Maxim Hendriks in Eindhoven did exactly this. He presented in his thesis [He13, p. 192–194] a curve $H \subset \mathbb{P}^6$ given as an intersection of 10 quadrics. Generators of the ideal defining H are

$$\begin{array}{l}
H:\\
-x_1x_2+x_1x_0+x_2x_6+x_3x_4-x_3x_5-x_3x_0-x_4x_6-x_5x_6,\\
x_1x_3+x_1x_6-x_2^2+2x_2x_5+x_2x_0-x_3^2+x_4x_5-x_4x_0-x_5^2,\\
x_1^2-x_1x_3+x_2^2-x_2x_4-x_2x_5-x_2x_0-x_3^2+x_3x_6+2x_5x_0-x_0^2,\\
x_1x_4-2x_1x_5+2x_1x_0-x_2x_6-x_3x_4-x_3x_5+x_5x_6+x_6x_0,\\
x_1^2-2x_1x_3-x_2^2-x_2x_4-x_2x_5+2x_2x_0+x_3^2+x_3x_6+x_4x_5+x_5^2-x_5x_0-x_6^2,\\
x_1x_2-x_1x_5-2x_1x_0+2x_2x_3-x_3x_0-x_5x_6+2x_6x_0,\\
-2x_1x_2-x_1x_4-x_1x_5+2x_1x_0+2x_2x_3-2x_3x_0+2x_5x_6-x_6x_0,\\
2x_1^2+x_1x_3-x_1x_6+3x_2x_0+x_4x_5-x_4x_0-x_5^2+x_6^2-x_0^2,\\
2x_1^2-x_1x_3+x_1x_6+x_2^2+x_2x_0+x_3^2-2x_3x_6+x_4x_5-x_4x_0+x_5^2-2x_5x_0+x_6^2+x_0^2,\\
x_1^2+x_1x_3-x_1x_6+2x_2x_5-3x_2x_0+2x_3x_6+x_4^2+x_4x_5-x_4x_0+x_6^2+3x_0^2.
\end{array}$$

Moreover Hendriks presents an explicit isomorphism between M and H (see also Theorem 1 below).

In §2.3 of a recent paper by Rubén Hidalgo [Hi15], another model over $\mathbb{Q}$ of the Fricke-Macbeath curve is mentioned. It is attributed to Bradley Brock, and given by the affine equation in two variables

$$1+7xy+21x^2y^2+35x^3y^3+28x^4y^4+2x^7+2y^7=0.$$

One readily calculates that this curve in $\mathbb{A}^2$ has as singularities 14 nodes, and its closure in $\mathbb{P}^2$ has no singular points at infinity. So indeed the equation defines a curve of genus 7. Using a basis of the regular 1-forms on the normalization, one obtains an embedding of the curve in $\mathbb{P}^6$. The resulting curve $B \subset \mathbb{P}^6$ can be given as follows (here and in other calculations Magma [BCP97] was used).

$$\begin{array}{l}
B:\\
x_0x_2+12x_3^2-x_4x_6,\\
-x_1^2+x_0x_3-2x_5x_6,\\
x_0x_4+16x_3x_5+8x_6^2,\\
-x_1x_3+x_0x_5+\frac{1}{2}x_2x_6,\\
-x_2x_3+2x_5^2+x_0x_6,\\
x_1x_2+12x_3x_5+4x_6^2,\\
-2x_2x_3+x_1x_4-8x_5^2,\\
-x_3^2+x_1x_5+\frac{1}{4}x_4x_6,\\
-\frac{1}{2}x_3x_4-\frac{1}{2}x_2x_5+x_1x_6,\\
x_2^2+2x_4x_5+8x_3x_6.
\end{array}$$

2. Results

First, we present explicit isomorphisms between the curves M, H, and B.

Theorem 1 (Hendriks, [He13])**.** *With notations as in the previous section and $\alpha := \zeta_7 + \zeta_7^{-1}$, an isomorphism $M \to H$ is given by $m \mapsto Am$, with $7A =$*

$$\begin{pmatrix} 0 & 0 & \alpha^2-\alpha-2 & -\alpha^2-\alpha-1 & 0 & 2\alpha^2-1 & 0 \\ \alpha^2-2\alpha & 0 & 0 & 0 & 3\alpha+1 & 0 & -\alpha^2-2\alpha+1 \\ 0 & 0 & -\alpha^2-\alpha-1 & -2\alpha^2+1 & 0 & \alpha^2-\alpha-2 & 0 \\ 0 & 7\alpha & 0 & 0 & 0 & 0 & 0 \\ -\alpha^2-2\alpha+1 & 0 & 0 & 0 & -\alpha^2+2\alpha & 0 & -3\alpha-1 \\ 0 & 0 & -3\alpha-1 & -\alpha^2+2\alpha & 0 & \alpha^2+2\alpha-1 & 0 \\ -3\alpha-1 & 0 & 0 & 0 & \alpha^2+2\alpha-1 & 0 & \alpha^2-2\alpha \end{pmatrix}.$$

Theorem 2. *With notations as in the previous section, an isomorphism $B \to H$ is given by $b \mapsto A'b$, with*

$$A' = \frac{1}{2}\begin{pmatrix} 2 & -8 & 4 & -24 & 1 & 24 & 0 \\ 2\sqrt{-7} & -4\sqrt{-7} & -2\sqrt{-7} & 0 & -\sqrt{-7} & 0 & -8\sqrt{-7} \\ 6 & 4 & -2 & -16 & 3 & 16 & 0 \\ 2\sqrt{-7} & -4\sqrt{-7} & -2\sqrt{-7} & -8\sqrt{-7} & -\sqrt{-7} & -8\sqrt{-7} & 16\sqrt{-7} \\ 0 & -4\sqrt{-7} & -2\sqrt{-7} & -8\sqrt{-7} & 0 & -8\sqrt{-7} & -8\sqrt{-7} \\ -2 & 8 & -4 & -32 & -1 & 32 & 0 \\ 2\sqrt{-7} & 0 & 0 & -8\sqrt{-7} & -\sqrt{-7} & -8\sqrt{-7} & -8\sqrt{-7} \end{pmatrix}.$$

Note that Theorems 1 and 2 imply that the three curves M, H, and B are isomorphic over $\mathbb{Q}(\zeta_7)$. Although both H and B are defined over $\mathbb{Q}$, they are not isomorphic over $\mathbb{Q}$. This follows, e.g., from the fact that both have good reduction modulo 3, and $\#H(\mathbb{F}_3) = 3 \neq 5 = \#B(\mathbb{F}_3)$.

From now on we focus on the model H presented by Hendriks. Our aim is to describe the jacobian $\mathrm{Jac}(H)$ up to isogenies defined over $\mathbb{Q}$, in terms of jacobians of certain quotients of H. To this end, let $X \subset \mathbb{P}^2$ be the plane quartic of genus 3 defined by

$$\begin{aligned} X: \quad & 5x^4 + 12x^3y + 6x^2y^2 - 4xy^3 + 4y^4 - 28x^3z + 16x^2yz - 24xy^2z + \\ & 16y^3z + 24x^2z^2 - 10y^2z^2 - 12xz^3 + 8yz^3 + 3z^4. \end{aligned}$$

Furthermore let E be the elliptic curve with equation

$$y^2 = x^3 + x^2 - 114x - 127.$$

The curve X defines the quotient of H by the involution $\mathrm{Diag}(-1,1,-1,1,1,-1,1)$. It is the image of H under $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_0 : x_2 : x_5)$. The elliptic curve E is obtained as a quotient of H by a group of order 7. Such a quotient was also described by Klaus Wohlfahrt in the corrigendum to his paper [Wo86]. His elliptic curve is in fact the quadratic twist by $\sqrt{-7}$ of E. The reader may verify that a very simple way to find the same elliptic curve as Wohlfahrt did, is by starting from the affine plane model of the Fricke-Macbeath curve given by Brock. Taking the quotient by $(x, y) \mapsto (\zeta_7 x, \zeta_7^{-1} y)$ yields Wohlfahrt's elliptic curve.

Theorem 3. *$Jac(H)$ is isogenous over $\mathbb{Q}$ to $Jac(X) \times Jac(X) \times E$.*

The next goal will be to analyse $\mathrm{Jac}(X)$. It turns out that $\mathrm{Aut}(X)$ contains a group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, with involutions defined over $\mathbb{Q}(\alpha)$. Moreover, these involutions are permuted by $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$. Let σ be a generator of this (cyclic) Galois group of order 3. The quotient of X by one of the involutions turns out to be a genus one curve C over $\mathbb{Q}(\alpha)$, with jacobian E' isogenous, again over $\mathbb{Q}(\alpha)$, to E. The action of σ yields the jacobians of the three quotients of X by the involutions. The restriction of scalars $\mathrm{Res}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(E')$, which is over $\mathbb{Q}(\alpha)$ isomorphic to $E' \times \sigma(E') \times \sigma^2(E')$, is the abelian threefold over $\mathbb{Q}$ we look for.

Theorem 4. *$Jac(X)$ is over $\mathbb{Q}$ isogenous to $Res_{\mathbb{Q}(\alpha)/\mathbb{Q}}(E')$, and the elliptic curves E and E' are isogenous over $\mathbb{Q}(\alpha)$.*

A straightforward consequence of Theorem 4 is a formula for $\#X(\mathbb{F}_q)$, for $q = p^n$ and p a prime $\neq 2, 7$:

Corollary 5. *The curve X has good reduction modulo every prime number $p \neq 2, 7$. If $q = p^n$ is a positive power of such a prime p, then*

$$\#X(\mathbb{F}_q) = \begin{cases} q + 1 & \text{if } q \not\equiv \pm 1 \bmod 7; \\ 3\#E(\mathbb{F}_q) - 2q - 2 & \text{if } q \equiv \pm 1 \bmod 7. \end{cases}$$

Combining Theorem 3 and Corollary 5 leads to the main result of this paper:

Theorem 6. *The curve H has good reduction modulo every prime number $p \neq 2, 7$. If $q = p^n$ is a positive power of such a prime p, then*

$$\#H(\mathbb{F}_q) = \begin{cases} \#E(\mathbb{F}_q) & \text{if } q \not\equiv \pm 1 \bmod 7; \\ 7\#E(\mathbb{F}_q) - 6q - 6 & \text{if } q \equiv \pm 1 \bmod 7. \end{cases}$$

The results described above are proven in the next section. In Section 4 we apply Theorem 6 to some particular prime powers q, resulting in various new records in the tables [GHLR09] maintained on `manypoints.org` of curves with many points over finite fields. In the same section we describe twists of $H/\mathbb{F}_q$ and we show examples where these lead to new records as well.

Most results of this paper were obtained during the master's project of the second author [V15], supervised by the first author.

3. Proofs

The statements in Theorem 1 and in Theorem 2 can be easily verified, so we omit this here. Instead, some comments are presented explaining how the isomorphisms were found. By construction, the curves M, H, and B are canonically embedded curves in $\mathbb{P}^6$. Hence an isomorphism between two of these curves is necessarily given by an element of $\mathrm{PGL}_7(\overline{\mathbb{Q}})$. Conjugation by this element then yields an isomorphism from the automorphism group of one curve to that of the other. By first determining such a conjugation, i.e., an $A \in \mathrm{PGL}_7(\overline{\mathbb{Q}})$ satisfying $A\alpha_1 = \alpha_2 A$ with α_1 running over the generators of some subgroup of the automorphisms of one curve, and the α_2 analogous generators of an isomorphic subgroup coming from the other curve, the isomorphisms were determined.

To make this explicit, consider the generators T, W of $\mathrm{Aut}(M) \subset \mathrm{PGL}_7(\mathbb{Q})$, defined as

$$T = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $T^3 = W^7 = (TW)^2 = id$. Corresponding generators R, S of $\mathrm{Aut}(H)$ satisfying $R^3 = S^7 = (SR)^2 = id$ one finds in the thesis of Hendriks [He13]. With $\alpha = \zeta_7 + \zeta_7^{-1}$

as before, they are $R :=$

$$\begin{pmatrix} 2\alpha^2+3\alpha-7 & 3\alpha^2+4\alpha+1 & 4\alpha^2+2\alpha-6 & -\alpha^2+\alpha+2 & -4\alpha^2-3\alpha+1 & -7\alpha^2-5\alpha+10 & 3\alpha^2-3\alpha-6 \\ 4\alpha^2+8\alpha-4 & 4\alpha^2+\alpha-11 & -9\alpha^2-3\alpha+14 & 2\alpha^2+\alpha-3 & -5\alpha^2-5\alpha+2 & -2\alpha^2-\alpha+3 & \alpha^2+6\alpha+5 \\ 2\alpha^2+4\alpha-2 & 6\alpha^2+3\alpha-9 & \alpha^2+\alpha-6 & -2\alpha^2-\alpha+3 & -\alpha^2+3\alpha-2 & -2\alpha^2-7\alpha+1 & -\alpha^2-4\alpha+5 \\ 14\alpha^2+7\alpha-21 & 7\alpha+7 & -7\alpha^2+7\alpha+14 & 0 & 7\alpha^2-7 & 0 & -7\alpha^2-7\alpha+14 \\ 6\alpha^2+9\alpha-7 & 5\alpha^2-\alpha-4 & -11\alpha^2-7\alpha+16 & -\alpha^2-2\alpha+1 & -3\alpha^2-4\alpha-1 & \alpha^2+2\alpha-1 & 5\alpha-3 \\ 6\alpha^2-10 & 3\alpha^2-5 & 4\alpha^2+8\alpha-4 & -\alpha^2-3 & -4\alpha^2+2 & -3\alpha^2-4\alpha-1 & 3\alpha^2-5 \\ 4\alpha^2+11\alpha-3 & 5\alpha^2-13 & -8\alpha^2-4\alpha+12 & -\alpha^2+\alpha+2 & -6\alpha^2-5\alpha+13 & \alpha^2-\alpha-2 & -\alpha^2+3\alpha-2 \end{pmatrix}$$

and $S :=$

$$\begin{pmatrix} -\alpha^2+4 & -\alpha-5 & 2\alpha^2+2\alpha-5 & 0 & -\alpha+2 & -2\alpha^2-\alpha+3 & -\alpha+2 \\ 3\alpha+1 & -\alpha^2-2\alpha+1 & -2\alpha^2-3\alpha & \alpha-2 & -3\alpha^2-\alpha+7 & \alpha^2+\alpha+1 & 3\alpha^2+3\alpha-4 \\ 2\alpha+3 & \alpha^2-4 & 2\alpha^2+\alpha-3 & 0 & \alpha^2+3 & -\alpha^2-2\alpha+1 & \alpha^2-4 \\ -7\alpha^2+14 & 0 & -7\alpha & 0 & 0 & -7 & 0 \\ -\alpha^2+2\alpha & -3\alpha-1 & -2\alpha^2-\alpha+3 & \alpha^2-4 & -\alpha^2+\alpha+2 & -\alpha+2 & 2\alpha^2+3\alpha \\ 2\alpha^2+3\alpha & 5\alpha^2+2\alpha-10 & -\alpha^2+\alpha+2 & 0 & -2\alpha^2-5\alpha+4 & -\alpha^2-\alpha-1 & -2\alpha^2+2\alpha+4 \\ \alpha^2+2\alpha-1 & \alpha^2-2\alpha & -3\alpha^2-3\alpha+4 & -\alpha^2-\alpha-1 & -3\alpha^2+5 & -\alpha^2+4 & 2\alpha^2+\alpha-3 \end{pmatrix}.$$

Solving for the matrix A in the linear equations $RA = AT, SA = AW$ then results in the desired isomorphism.

In the case of the curves B and H, the only obvious automorphisms of B form a dihedral group of order 14. On the plane model, this group is generated by $(x, y) \mapsto (y, x)$ and $(x, y) \mapsto (\zeta_7 x, \zeta_7^{-1} y)$. On B this yields the matrices

$$D := \mathrm{Diag}(\zeta_7^5, \zeta_7^3, \zeta_7^4, \zeta_7, \zeta_7^2, \zeta_7^6, 1)$$

and

$$F := \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A corresponding dihedral group in $\mathrm{Aut}(H)$ is the one with generators

$$\tau := (S^{-1}RS^{-1})^2 RS = \mathrm{Diag}(-1, 1, -1, 1, 1, -1, 1)$$

and $L := (S^{-2}R)^2 S^{-1}$, given by $14L :=$

$$\begin{pmatrix} -4\alpha^2-\alpha+4 & 3\alpha^2-12 & 4\alpha^2+2\alpha-6 & -\alpha^2+\alpha+2 & 5\alpha+4 & \alpha^2+5\alpha & -5\alpha^2-3\alpha+12 \\ -2\alpha^2-6\alpha+6 & -3\alpha-1 & 7\alpha^2+7\alpha-7 & -\alpha-5 & -7\alpha^2-3\alpha+13 & -2\alpha^2+\alpha-1 & 7\alpha^2+4\alpha-8 \\ 2\alpha^2+4\alpha-2 & -2\alpha^2-5\alpha-3 & 5\alpha^2-\alpha-11 & -2\alpha^2-\alpha+3 & 3\alpha^2+3\alpha+3 & -4\alpha^2+\alpha+7 & -5\alpha^2+6 \\ -7\alpha & -7\alpha-7 & 7\alpha^2+7\alpha-7 & 7 & -7\alpha^2+7 & -7 & 7\alpha^2+7\alpha-14 \\ -7\alpha & -3\alpha^2-7\alpha+5 & \alpha^2+3\alpha-3 & -\alpha^2-3 & -3\alpha^2+5 & 3\alpha^2+2\alpha-9 & 4\alpha^2+7\alpha-2 \\ 4\alpha^2+8\alpha-4 & 5\alpha^2-4\alpha-12 & -4\alpha^2-2\alpha+6 & -\alpha^2+4 & -6\alpha^2-2\alpha+14 & -\alpha^2+2\alpha & -\alpha^2+6\alpha+6 \\ 2\alpha^2-\alpha-6 & 3\alpha^2-4\alpha-4 & 6\alpha^2+4\alpha-4 & \alpha^2+\alpha-6 & -4\alpha^2+3\alpha+10 & -\alpha^2-3\alpha-4 & 3\alpha^2+3\alpha-4 \end{pmatrix}.$$

Solving the system $\tau A' = A'F$, $LA' = A'D$ for the matrix A' yields the map given in Theorem 2.

We will now prove Theorem 3. Denote $\pi : (x_0 : \ldots : x_6) \mapsto (x_0 : x_2 : x_5)$, with $\pi(H) = X$. Let $\omega_1, \omega_2, \omega_3$ be a basis for the space of regular differentials $H^0(X, \Omega^1)$. A calculation shows that $\pi^*\omega_1, \pi^*\omega_2, \pi^*\omega_3, \tau^*\pi^*\omega_1, \tau^*\pi^*\omega_2, \tau^*\pi^*\omega_3$ are linearly independent in $H^0(H, \Omega^1)$. Hence

$$(\pi, \pi\tau) \; : \; H \to X \times X$$

induces a homomorphism $\mathrm{Jac}(X) \times \mathrm{Jac}(X) \to \mathrm{Jac}(H)$ with finite kernel. Then the cokernel is an abelian variety of dimension 1 defined over $\mathbb{Q}$, i.e., it is an elliptic curve over $\mathbb{Q}$. We briefly sketch two methods to find an equation for this elliptic curve (in the first case, up to isogeny over $\mathbb{Q}$).

For the first method, observe that the curve B (and hence also H) has good reduction except at the primes 2 and 7. A convenient way to verify this, is by using the plane model of B. A consequence of this is, that the desired elliptic curve has good reduction away from 2 and 7. So its conductor divides $2^8 \cdot 7^2$. Moreover, by construction the number of rational points on this elliptic curve over $\mathbb{F}_p$ for a prime $p \neq 2, 7$ equals

$$\#H(\mathbb{F}_p) - 2\#X(\mathbb{F}_p) + 2p + 2.$$

This information suffices to determine the correct isogeny class among the finitely many possible ones.

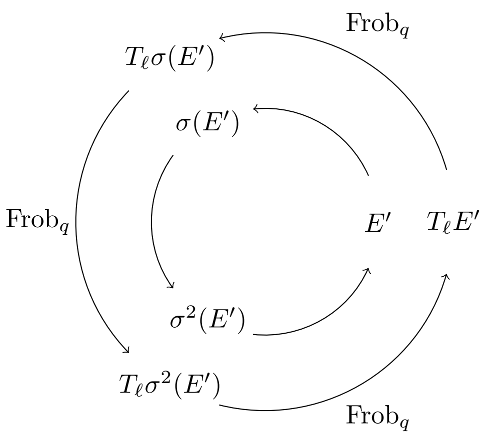
Alternatively, and more geometrically, take $\rho := (SR^{-1}S)^3$ which is an automorphism defined over the ground field, of order 3. A calculation (compare [V15, p. 13-15]) reveals that $H/\langle\rho\rangle$ is a curve of genus 1, given by $y^2 = -7t^4 - 28t^3 - 56t^2 - 28$. The jacobian of this curve is our curve E. Since ρ^* fixes no differentials in the subspace of $H^0(X, \Omega^1)$ spanned by $\pi^*\omega_1, \pi^*\omega_2, \pi^*\omega_3, \tau^*\pi^*\omega_1, \tau^*\pi^*\omega_2, \tau^*\pi^*\omega_3$, it follows that $\mathrm{Jac}(H) \sim \mathrm{Jac}(X) \times \mathrm{Jac}(X) \times E$.

Next we prove Theorem 4. For this, one observes that X admits over $\mathbb{Q}(\alpha)$ the involution given by

$$A := \frac{1}{7}\begin{pmatrix} -4\alpha^2 - 4\alpha + 3 & -2\alpha^2 + 2\alpha + 4 & 4\alpha^2 + 2\alpha - 6 \\ -4\alpha^2 - 2\alpha + 6 & 2\alpha^2 + 2\alpha - 5 & 2\alpha^2 + 4\alpha - 2 \\ 2\alpha^2 - 2\alpha - 4 & 6\alpha + 2 & 2\alpha^2 + 2\alpha - 5 \end{pmatrix}.$$

Moreover, A and its conjugates $\sigma(A)$ and $\sigma^2(A)$ generate a group of automorphisms of X isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The quotient of X by any nontrivial element of this group is an elliptic curve over $\mathbb{Q}(\alpha)$, and the three elliptic curves obtained in this way are obviously conjugate. There are no nontrivial regular differentials on X fixed by all three involutions. This suffices to conclude that $\mathrm{Jac}(X)$ is isogenous over $\mathbb{Q}$ to the restriction of scalars of any of the three elliptic curves. The elliptic curve E has its three points of order 2 defined over $\mathbb{Q}(\alpha)$. The 2-isogenies resulting from this, turn out to have as image curves exactly the three elliptic curves we found as quotients of X. This proves Theorem 4.

Corollary 5 is an immediate consequence of Theorem 4. The statement concerning good reduction is easily verified. In the case $q \equiv \pm 1 \bmod 7$, all 7-th roots of unity exist in $\mathbb{F}_{q^2}$ and hence $\mathbb{F}_q$ contains the residue class field at the primes dividing q of $\mathbb{Q}(\alpha)$. As a consequence, $\mathrm{Jac}(X)$ is isogenous over $\mathbb{F}_q$ to $E \times E \times E$, from which the formula for the number of points in this case is immediate. For $q \not\equiv \pm 1 \bmod 7$, the residue class field of $\mathbb{Q}(\alpha)$ at primes dividing q is not contained in $\mathbb{F}_q$ but in its cubic extension. Hence the q-th power Frobenius permutes the reductions of the three curves $E', \sigma(E')$, and $\sigma^2(E')$. This implies that the trace of Frobenius on $\mathrm{Res}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(E')$ is zero, implying the remaining formula.



This completes the proof of the results presented in Section 2.

4. Examples and twists

The website `http://manypoints.org/` [GHLR09] lists, for small genera g and small cardinalities q of a finite field, an upper bound up for the cardinality $\#C(\mathbb{F}_q)$ of any smooth, complete and absolutely irreducible curve C of genus g defined over $\mathbb{F}_q$. In many instances this is the Hasse-Weil-Serre bound $q+1+gm$, in which m is the largest integer $\leq \sqrt{4q}$. In case a curve reaching this bound is known to exist, this means the number $N_q(g)$, denoting the maximum over all such cardinalities $\#C(\mathbb{F}_q)$ for fixed g, q is determined. If no curve reaching the upper bound is known then the tables aim to list an example with at least $\mathtt{up}/\sqrt{2}$ rational points. We now list the cases in which the curve H provides such an example. Instances where an example with at least as many points is known, will be ignored. Somewhat surprisingly, even with $q \not\equiv \pm 1 \bmod 7$, in which case Theorem 6 shows that H has (only) as many rational points as the elliptic curve E, some new entries were found.

q	3^3	5^3	5^5	11^5	13^2	13^3	13^4	13^5
up	95	277	3903	166666	352	2849	30928	379820
$\lceil \mathtt{up}/\sqrt{2} \rceil$	68	196	2760	117851	249	2015	21870	268574
$\#H(\mathbb{F}_q)$	84	252	3183	161625	324	2688	27540(∗)	362880(∗)

q	17^3	17^5	19^4	19^5	29	97
up	5892	1436539	135376	2498129	100	231
$\lceil \mathtt{up}/\sqrt{2} \rceil$	4167	1015787	95726	1766444	71	164
$\#H(\mathbb{F}_q)$	5796	1417575	129675(∗)	2477811	72	168

In a sense the 'smallest' example here is $\#H(\mathbb{F}_{27}) = 84$. The previous record for $q = 27$ and $g = 7$ was obtained by Sémirat [Sém00] in 2000, who found an example having 82 rational points. The three marked (∗) entries show examples which we will improve now, as follows.

A natural attempt to obtain more examples with many points from the curve H, is to consider twists of it over $\mathbb{F}_q$, i.e., curves over the same field which are isomorphic to H over some extension field. We refer to [MT] for some general theory concerning twists. The twists over $\mathbb{F}_q$ are in $1-1$ correspondence with the set $H^1(\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q), \mathrm{Aut}(H))$, and the latter set allows a natural bijection to the set of 'Frobenius conjugacy classes' in $\mathrm{Aut}(H)$.

In our case, we consider $H^1(\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q), G)$, with $G \subset \mathrm{PGL}_7(\overline{\mathbb{F}_q})$ the simple group of order 504 acting as automorphisms on H. These automorphisms are defined over $\mathbb{F}_q(\zeta+\zeta^{-1})$ with ζ a primitive 7th root of unity. Hence the Galois action on G is trivial precisely when $q \equiv \pm 1 \bmod 7$. In this case, Frobenius conjugacy classes coincide with ordinary conjugacy classes, and there are 9 of these. For $q \not\equiv \pm 1 \bmod 7$ a calculation with Magma shows that there exist only 3 Frobenius conjugacy classes.

If an automorphism β defines some Frobenius conjugacy class, then the corresponding cocycle class is represented by the cocycle defined by $\mathrm{Frob}_q \mapsto \beta$. It defines a twist H^{tw}, and by construction rational points on this twist correspond to points $P \in H(\overline{\mathbb{F}_q})$ such that $\beta(\mathrm{Frob}_q(P)) = P$. This allows one to calculate, for given q and β, the number of rational points $\#H^{\mathrm{tw}}(\mathbb{F}_q)$.

Ignoring the trivial twist which results in the curve H itself, we can improve 3 of the records presented in the earlier table. They are given below.

(1) $q = 13^4 \equiv 1 \bmod 7$. The (cubic) twist corresponding to $\mathrm{Frob}_q \mapsto R$ has 28854 rational points. This exceeds $\#H(\mathbb{F}_{13^4}) = 27540$.
(2) $q = 13^5 \equiv -1 \bmod 7$. The quadratic twist corresponding to the cocycle $\mathrm{Frob}_q \mapsto \tau = (S^{-1}RS^{-1})^2RS$ has 372496 rational points, while $\#H(\mathbb{F}_{13^5}) = 362880$.

(3) (Again) $q = 13^5 \equiv -1 \bmod 7$. The cubic twist corresponding to the cocycle $\mathrm{Frob}_q \mapsto R$ has 373698 rational points, improving what was found in (2) above.

(4) $q = 19^4 \equiv 2 \bmod 7$. Here, the quadratic twist corresponding to the cocycle $\mathrm{Frob}_q \mapsto \tau = (S^{-1}RS^{-1})^2RS$ has 130969 rational points, whereas $\#H(\mathbb{F}_{19^4}) = 129675$.

Using the `Sage` code from appendix A.2.3 of [V15] one can calculate explicit models for the desired twists. As an example, the quadratic twist using the automorphism $(x, y) \mapsto (y, x)$ of the affine curve $1+7xy+21x^2y^2+35x^3y^3+28x^4y^4+2x^7+2y^7 = 0$ the twist over $\mathbb{F}_q(\sqrt{d})/\mathbb{F}_q$ is given by:

$$7d^4x^8 + 2d^4x^7 - 28d^3x^6y^2 + 35d^4x^6 + 42d^3x^5y^2 + 42d^2x^4y^4 - 105d^3x^4y^2 \\ +70d^2x^3y^4 - 28dx^2y^6 + 84d^4x^4 + 105d^2x^2y^4 + 14dxy^6 + 7y^8 - 168d^3x^2y^2 - 35dy^6 \\ +112d^4x^2 + 84d^2y^4 - 112d^3y^2 + 64d^4 = 0.$$

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